APPROXIMATE AVERAGING OF THE HEAT-CONDUCTION EQUATION

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Simple formulas are obtained for calculating the mean nonstationary temperature of an arbitrary fuel element. The proposed "mean adjoint" approximation is found to have definite advantages over the quasistationary approximation.

Thermophysical calculations frequently require the solution of the heat-conduction equation*

$$c\gamma \frac{\partial t}{\partial \tau} = \lambda \Delta t + q \tag{1}$$

for a uniform fuel element with a convection boundary condition

$$-\lambda \, \mathrm{grad}_n t(S, \tau) = \alpha t(S, \tau) \tag{2}$$

and some initial conditions

$$t_0 = t(x, \tau). \tag{3}$$

An analytic solution of the boundary-value problem (1)-(3) is known only for very simple cases [1].

In many engineering calculations it is sufficient to find the change in the mean temperature of the fuel element

$$\bar{t} = \frac{\int t dV}{\int dV} \tag{4}$$

as a function of the time; $\bar{t} = \bar{t}(\tau)$. This, together with the energy balance, permits a calculation of the wall temperature $t(\bar{S}, \tau)$, and the set of equations describing the nonstationary process becomes complete.

The functions in Eqs. (1)-(3), defined on the interval $\tau_0 \leq \tau < \infty$, can always be continued into the interval $-\infty < \tau < \infty$ in such a way as to satisfy the condition

$$\bar{t}(-\infty) = 0. \tag{5}$$

This permits a solution which is linearly dependent on q. In addition we assume that the whole time interval can be divided into a number of intervals in each of which the thermophysical parameters are approximately constant. Then the solution of the problem in one interval is the initial condition for the next.

Quasistationary Approximation. Suppose that after a sufficiently long time a steady heat release rate is established in the fuel element, i.e.

$$\frac{\partial q}{\partial \tau}(x, \infty) = 0. \tag{6}$$

Then the solution of the boundary-value problem (1)-(3) will asymptotically approach the solution of the stationary equation

$$0 = \lambda \Delta t^{st} + q(x, \infty); \quad -\lambda \, \mathrm{grad}_n t^{st}(S) = \alpha t^{st}(S), \tag{7}$$

*In Eqs. (1)-(3) all temperatures are measured from the temperature of the coolant.

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which, after some simple transformations, can be integrated to give

$$q' = \frac{\int q(x, \infty)\, dV}{\int dV} = \alpha \frac{\int t^{st}(S)\, dS}{\int dV}. \tag{8}$$

In the quasistationary or parabolic approximation [2] it is assumed that the nonstationary temperature is proportional to the stationary: $t(x, \tau) \approx t^{st}(x) a(\tau)$. The stationary equation (7) is approximated by the equation

$$0 = \lambda \Delta t' + q'; \quad -\lambda\, \mathrm{grad}_n t'(S) = \alpha t'(S). \tag{9}$$

In order that a solution of the form $t'(x) = \varphi(x) q'$ can always be obtained we must have

$$\lambda \Delta \varphi = -1; \quad -\lambda\, \mathrm{grad}_n \varphi(S) = \alpha \varphi(S). \tag{10}$$

Thus in the quasistationary approximation it is assumed that

$$t(x, \tau) \approx t^{st}(x) \;\; a(\tau) \approx q' \varphi(x) a(\tau). \tag{11}$$

Substituting (11) into both sides of Eq. (8) and using (4) leads to

$$\alpha \frac{\int t(S, \tau)\, dS}{\int dV} = \frac{\bar{t}}{\varphi}. \tag{12}$$

Using this in averaging the heat-conduction equation (1) gives an equation for the approximate average temperature of the fuel element

$$c\gamma \frac{\partial \bar{t}}{\partial \tau} = \frac{\bar{t}}{\bar{\varphi}} + \bar{q} \tag{13}$$

with the condition (5). It is easy to find a solution of these equations

$$\bar{t}(\tau) = \frac{1}{c\gamma} \int_{-\infty}^{\tau} \exp[\sigma(\tau' - \tau)]\, \bar{q}(\tau')\, d\tau', \tag{14}$$

where

$$\sigma = \frac{1}{c\gamma\bar{\varphi}}; \quad \bar{\varphi} = \frac{\int \varphi dV}{\int dV}; \quad \bar{q} = \frac{\int q dV}{\int dV}, \tag{15}$$

and $\bar{q}(\tau)$ is continued into the interval $\tau < \tau_0$ in such a way as to satisfy the initial condition

$$\bar{t}_0 = \frac{1}{c\gamma} \int_{-\infty}^{\tau_0} \exp[\sigma(\tau' - \tau_0)]\, \bar{q}(\tau')\, d\tau'. \tag{16}$$

Adjoint Temperature Averaging. In order to obtain a more accurate formula for the mean temperature of a fuel element we use the results of [3]. For our purposes it is sufficient to understand by the adjoint of t the function t^* satisfying

$$\int (t \Delta t^* - t^* \Delta t)\, dV = 0, \tag{17}$$

where the integration is extended over the whole volume of the fuel element.

We write the stationary equation adjoint to the initial boundary-value problem (1)-(3) in the form

$$0 = \lambda \Delta t^* + q^*; \quad -\lambda\, \mathrm{grad}_n t^*(S, \tau) = \alpha t^*(S, \tau), \tag{18}$$

where it is assumed that the fictitious source $q^* = q^*(\tau)$ depends only on the time. The boundary conditions (2) and (18) ensure that t and t^* are adjoints, as is easy to see by applying Green's theorem to Eq. (17) [3].

We now multiply Eq. (1) by t^* and (18) by t, subtract one from the other, and integrate over the volume of the fuel element. Using (17) we obtain

$$c\gamma \int t^* \frac{\partial t}{\partial \tau} dV = \int q t^* dV - q^* \int t dV. \tag{19}$$

Since the average temperature of the fuel element is required, it is natural to demand that the fictitious source $q^* = q^*(\tau)$ ensure that

$$\int t^* dV = \int t dV, \tag{20}$$

which implies that the mean temperature of the fuel element equals the mean adjoint temperature

$$\bar{t}(\tau) = \bar{t}^*(\tau). \tag{21}$$

In this case Eq. (19) is reduced to the form

$$c\gamma \frac{\widetilde{\partial t}}{\partial \tau} = \tilde{q} - q^*, \tag{22}$$

where the tilde denotes adjoint temperature averaging:

$$\tilde{q} = \frac{\int q t^* dV}{\int t^* dV}. \tag{23}$$

Equation (18) can be solved in the same way as (9)

$$t^*(x, \tau) = \varphi(x) q^*(\tau), \tag{24}$$

where the spatial distribution $\varphi(x)$ is given by (10). If $q^*(\tau)$ were known, the determination of the mean adjoint temperature would solve the problem according to (21). Unfortunately it is necessary to make an approximation in order to find q^* from Eq. (22).

We write the solution of the boundary-value problem (1)-(3) in the form $t = t^* + \theta$. It then follows from (21) that the mean deviation of the nonstationary temperature from the adjoint must be zero: $\bar{\theta} = 0$. Therefore it seems very plausible to assume that the adjoint temperature average of the deviation $\tilde{\theta}$ is also changed insignificantly, i.e.

$$\frac{\widetilde{\partial \theta}}{\partial \tau} = \frac{\int t^* \frac{\partial \theta}{\partial \tau} dV}{\int t^* dV} \approx 0. \tag{25}$$

The calculations performed above can be made by using the apparatus of adjoint functions if we set $q^*(\tau) = q' a(\tau)$. Then the quasistationary approximation (11), even neglecting the error in approximating Eq. (7) by (9), can be assumed in our notation

$$t(x, \tau) \approx t^*(x, \tau); \quad \theta(x, \tau) \approx 0. \tag{26}$$

Thus the mean adjoint approximation (25) guarantees an accuracy which is at least no worse than the quasistationary approximation.

By using (24) Eq. (22) in the mean adjoint approximation is reduced to the form

$$\frac{\partial q^*}{\partial \tau} = \mu(\tilde{q} - q^*), \tag{27}$$

where

$$\mu = \frac{1}{c\gamma\tilde{\varphi}}; \quad \tilde{\varphi} = \frac{\int \varphi^2 dV}{\int \varphi dV}; \quad \tilde{q} = \frac{\int \varphi q dV}{\int \varphi dV}. \tag{28}$$

Keeping in mind condition (5), integrating (27) and substituting the result into (24) we obtain by using (21)

$$\bar{t}(\tau) = \mu\bar{\varphi} \int_{-\infty}^{\tau} \exp[\mu(\tau' - \tau)] \tilde{q}(\tau') d\tau', \tag{29}$$

where $\tilde{q}(\tau)$ is continued into $\tau < \tau_0$ in such a way as to satisfy the initial condition

$$\bar{t}_0 = \mu\bar{\varphi} \int_{-\infty}^{\tau_0} \exp[\mu(\tau' - \tau_0)] \tilde{q}(\tau') d\tau'. \tag{30}$$

Comparing (14) and (29) we note that these formulas for the mean temperature of the fuel element differ in the nature of the averaging and almost coincide when the stationary temperature distribution is nearly uniform ($\varphi(x) \approx$ const).

<u>One-Dimensional Problems.</u> If the fuel element has a simple geometry the stationary equation (10) is easy to solve, and this permits an easy calculation of all the quantities appearing in Eqs. (16) and (29) for the

mean temperature of the fuel element. In particular for one-dimensional problems (solid plate, cylinder, and sphere) the solution of Eq. (10) can be expressed by the single formula:

$$\varphi(x) = \frac{1}{k}\frac{L^2}{\lambda}\psi(x), \tag{31}$$

where

$$\psi(x) = \frac{1}{2}\left[1 - \left(\frac{x}{L}\right)^2 + \frac{2}{\mathrm{Bi}}\right], \tag{32}$$

and the characteristic number k and the characteristic dimension L are given by the specific geometry:

Fuel Element	k	L
Plate	1	Half-thickness
Cylinder	2	Radius
Sphere	3	Radius

The substitution of (31) gives the following expressions for the mean temperature of the fuel element:

$$\bar{t}_{\mathrm{q.s}}(\mathrm{Fo}) = \frac{L^2}{\lambda}\int_{-\infty}^{\mathrm{Fo}} \exp\left[\frac{k}{\bar{\psi}}(\mathrm{Fo}' - \mathrm{Fo})\right]\bar{q}(\mathrm{Fo}')\,d\mathrm{Fo}'; \tag{33}$$

$$\bar{t}_{\mathrm{m.a}}(\mathrm{Fo}) = \frac{L^2}{\lambda}\frac{\bar{\psi}}{\tilde{\psi}}\int_{-\infty}^{\mathrm{Fo}} \exp\left[\frac{k}{\tilde{\psi}}(\mathrm{Fo}' - \mathrm{Fo})\right]\tilde{q}(\mathrm{Fo}')\,d\mathrm{Fo}' \tag{34}$$

for the quasistationary and mean adjoint approximations respectively. The averages $\bar{\psi}$ and $\tilde{\psi}$ entering (33) and (34) are calculated by integrating over the volume of the fuel element in the appropriate coordinate system. The results can again be written in a single form:

$$\bar{\psi} = \frac{1}{\mathrm{Bi}} + \frac{1}{k+2}; \tag{35}$$

$$\tilde{\psi} = \frac{1}{\bar{\psi}}\left[\frac{1}{\mathrm{Bi}^2} + \frac{2}{k+2}\frac{1}{\mathrm{Bi}} + \frac{2}{(k+2)(k+4)}\right]. \tag{36}$$

<u>Numerical Example.</u> In the simplest case when

$$q = \begin{cases} 0; & \tau < 0, \\ \mathrm{const}; & \tau > 0, \end{cases} \qquad \bar{t}_0 = 0, \tag{37}$$

Eqs. (33)-(36) for a cylindrical fuel element take the form

$$\bar{t}_{\mathrm{q.s}} = \frac{qR^2}{2\lambda}\bar{\psi}\left[1 - \exp\left(-\frac{2}{\bar{\psi}}\mathrm{Fo}\right)\right]; \tag{38}$$

$$\bar{t}_{\mathrm{m.a}} = \frac{qR^2}{2\lambda}\bar{\psi}\left[1 - \exp\left(-\frac{2}{\tilde{\psi}}\mathrm{Fo}\right)\right]; \tag{39}$$

$$\bar{\psi} = \frac{1}{\mathrm{Bi}} + \frac{1}{4}; \quad \tilde{\psi} = \frac{\dfrac{1}{\mathrm{Bi}^2} + \dfrac{1}{2\mathrm{Bi}} + \dfrac{1}{12}}{\dfrac{1}{\mathrm{Bi}} + \dfrac{1}{4}}. \tag{40}$$

At the same time the exact solution of the case under consideration is known [1]. In our notation it has the form

$$\bar{T} = \frac{qR^2}{2\lambda}\left[\bar{\psi} - 2\lambda\sum_{n=1}^{\infty}\frac{B_n}{\mu_n^2}\exp(-\mu_n^2\mathrm{Fo})\right], \tag{41}$$

with the coefficients given by

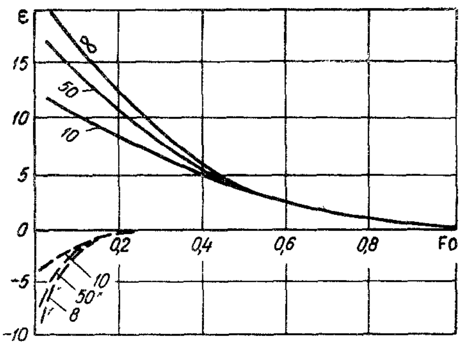

Fig. 1. Calculational errors. The numbers on the curves are values of the Biot number. $\varepsilon = [(\bar{t}/T) - 1]$ in % and $\mathrm{Fo} = (a/R^2)\tau$.

$$B_n = \frac{4\mathrm{Bi}}{\mu_n^2(\mu_n^2 + \mathrm{Bi}^2)}, \tag{42}$$

where the roots μ_n are found from the transcendental equations

$$\mu_n = \frac{J_0(\mu_n)}{J_1(\mu_n)}\,\mathrm{Bi}. \tag{43}$$

The approximate formulas have advantages in making calculations even when an exact solution exists. The only question concerns the accuracy.

Figure 1 shows the errors in calculating the mean temperature of a cylindrical fuel element in the quasistationary (solid curves) and mean adjoint (open curves) approximations as functions of the Fourier and Biot numbers. The exact solution was taken as Eq. (41), breaking off the infinite series after the first six terms [1].

Figure 1 shows that the quasistationary approximation overestimates the mean fuel element temperature, while the mean adjoint approximation underestimates it. As regards the accuracy of the calculation, our method leads to an appreciably smaller error and converges appreciably more rapidly to the exact solution. For Bi = 10 the mean adjoint approximation is accurate to within 3% from the instant when the temperature reaches a level $0.1\, t^{st}$, while the quasistationary approximation reaches this accuracy only after $t \gtrsim 0.9\, t^{st}$.

In addition it should be noted that for nonuniform heat release ($q(x) \neq \mathrm{const}$) the error in the quasistationary approximation is increased further because of the error in approximating the stationary Eq. (7) by Eq. (9) with an average heat release.

NOTATION

c	is the specific heat of the fuel element;
γ	is the density;
λ	is the thermal conductivity;
α	is the heat transfer coefficient;
t	is the temperature;
τ	is the time;
Δ	is the Laplacian operator;
q	is the volumetric thermal source strength;
S	is the boundary of the fuel element;
t^{st}	is the stationary temperature;
$*$	denotes a parameter of the adjoint equation;
$\varphi(x)$	is the spatial distribution of the stationary temperature;
$\underline{f}$	is the mean integral of $f(x)$;
$\tilde{f}$	is the mean adjoint of $f(x)$.

LITERATURE CITED

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